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STRESS INTENSITY FACTOR CALCULATIONS BASED ON A CONSERVATION INTEGRAL

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Abstract—It is observed that one of the integral conservation laws of elastostatics, the so-called M-integral conservation law, has certain special features which make it possible to apply this conservation law for a class of plane elastic crack problems in order to calculate the elastic stress intensity factor in each case without solving the corresponding boundary value problem. The main characteristics which a problem must have in order for the approach to be useful are (1) for points very near to the origin of coordinates, the known elastic stresses are $O(r^{-\gamma})$ where r is the radial coordinate and $\gamma \leq 1$, (2) for points very far from the origin, the known elastic stresses are $O(r^{-\gamma})$ where $\gamma \geq 1$, and (3) the boundary of the body is made up of radial lines on which certain traction and/or displacement conditions are satisfied. The approach is demonstrated by determining the stress intensity factors for four familiar elastic crack problems directly from the conservation law, and then four similar additional applications of the M-integral conservation law are discussed.

INTRODUCTION

Some conservation laws applicable in elastostatics were recently discovered by Knowles and Sternberg[1] who applied Noether's theorem on invariant variational principles in conjunction with the stationary potential energy principle of elastostatics. The mathematical statement of each conservation law is that the integral of a certain functional of the elastic field over the bounding surface of a regular closed subregion of the deformed elastic solid equals zero. According to Eshelby[2], Günther[3] had previously obtained the same set of elastostatic conservation laws. In a subsequent paper, Budiansky and Rice[4] considered the surface integrals which appear in the conservation laws over the boundary of a subregion of the body which contains a cavity. They showed that in this case the surface integrals were nonzero and, in fact, that they could be interpreted as potential energy release rates associated with translation, rotation and expansion of the cavity. In this paper, certain special features of one of the conservation integrals, the so-called M-integral, are exploited to determine stress intensity factors for a variety of elastic crack problems.

THE M-INTEGRAL

General properties

Consider an elastic solid which is in a state of plane two-dimensional deformation. The displacement vector u_i depends only on the rectangular Cartesian coordinates x_1 and x_2. For any simple curve C in the plane of deformation, the value of M is defined by the line integral

$$M = \int_C (Wn_ix_i - T_ku_{k,i}x_i)\,\mathrm{d}S \tag{1}$$

where W is the elastic energy density, n_i is the unit normal to C which (by convention) is directed to the right as C is traversed in a specified direction, and T_i is the traction acting on the material to the left of C. The elastic energy W depends on the strain $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$ and the stress $\sigma_{ij} = \sigma_{ji}$, which is related to the traction on C according to $\sigma_{ij}n_i = T_j$, satisfies the stress–strain relation $\sigma_{ij} = \partial W/\partial\epsilon_{ij}$ and the equilibrium equation $\sigma_{ij,j} = 0$ at all points of the body. As the form of the equilibrium equation implies, body forces are not considered. It can be shown (see, e.g. [1, 2]) that if the material is homogeneous and if $2W = \sigma_{ij}u_{i,j}$ then $M = 0$ for any *closed* contour C which bounds a simply-connected region of the body. Thus, $M = 0$ is one of the so-called conservation laws of elastostatics[1] and it applies for homogeneous linear elastic materials which are in a state of plane infinitesimal deformation. The material need not be isotropic, although material isotropy will be assumed for all examples to be considered here.

The property of path-independence of M in (1) follows directly from the conservation law.

Consider two paths C_1 and C_2 in the body with common end-points and which together enclose a simply-connected region, but which are otherwise arbitrary. The sense of traversal is the same for the two paths, that is, from one common end-point to the other. The conservation law implies that $M = 0$ for the closed path $C_1 - C_2$, where the minus sign indicates that part of the closed contour (in this case C_2) is traversed in a negative sense. It follows immediately from a fundamental property of line integrals that if $M(C_1 - C_2) = 0$ then $M(C_1) = M(C_2)$. Therefore, M is a path-independent line integral. If the region enclosed by a contour C contains a cavity then the value of M for this contour is nonzero, in general. Budiansky and Rice[4] have shown that the value of M is the rate of potential energy release for self-similar expansion of the cavity in this case. Actually, M is a measure of energy release rate with respect to the relative scale change $\mathrm{d}L/L$, where L is any characteristic length of the cavity.

In this study, the interpretation of the M-integral as an energy release rate does not play a significant role. The property that $M = 0$ for any closed contour is here exploited to determine stress intensity factor solutions for a number of cracked elastic solid configurations of interest in fracture mechanics. It will be shown that, for a certain class of elastic crack problems, the M-integral has several special features which make it possible to relate the crack tip stress intensity factor to the applied loading without actually solving the corresponding elastostatic boundary value problem. The approach requires a choice of contour C such that contributions to M from the various segments of C can be calculated directly in terms of either the stress intensity factor or the imposed load and constraint conditions. Rice has made similar use of the J-integral on pp. 231–232 of [5], and some of the applications of the M-integral to be discussed here have been anticipated by Eshelby[2]. In the next section, a few of the key features of the M-integral are noted. The general approach is then demonstrated by calculating stress intensity factors for a number of crack problems which have been analyzed previously by more traditional methods and finally the results for a number of previously unsolved problems are determined.

Special features

Suppose that the two-dimensional body under consideration contains a crack, and that the coordinates of a particular crack tip are y_1, y_2 with $y^2 = y_i y_i > 0$. The contribution to M from a vanishingly small path which begins on one crack face, surrounds the crack tip, and terminates on the opposite face is $y_i J_i$ where J_i is the vector line integral

$$J_i = \int_C (W n_i - T_k u_{k,i})\, \mathrm{d}s. \qquad (2)$$

Furthermore, if the crack faces coincide with radial lines in the coordinate system being used then the value of $y_i J_i$ is simply $\pm yJ$ where the algebraic sign is determined from the details of the problem and J is Rice's integral of fracture mechanics[5]. It was shown by Rice[5] that J is independent of path for paths which originate on one traction-free crack face, surround the tip, and terminate on the opposite traction-free face. Furthermore, J is equal to Irwin's potential energy release rate G for coplanar crack growth and is therefore related to the plane strain mode I and mode II stress intensity factors through Irwin's relationship[6]

$$J = G = \frac{1 - \nu^2}{E}[K_I^2 + K_{II}^2] \qquad (3)$$

where E and ν are Young's modulus and Poisson's ratio for the isotropic elastic material. For the special case $y = 0$, the crack tip is at the origin and there is no contribution to M from a vanishingly small path surrounding the tip. Eshelby[2] noted that this last feature of M could be used to advantage in certain cases, as will be seen in the subsequent discussion of example problems.

It is clear from (1) that the first term in the integrand of the M-integral vanishes on any part of C which coincides with a segment of a radial line in the coordinate system used. At any point on a radial segment, the normal n_i is perpendicular to the position vector which has components x_i, so that $x_i n_i = 0$ there. The second term in the integrand of the M-integral also vanishes on a part of C which coincides with a segment of a radial line under certain conditions. For example, this term is identically zero on a radial segment if the traction T_i vanishes there, if the displacement is uniform so that $u_{i,j}x_j = 0$ there, or if a component of the traction vanishes and the complimentary component of displacement is uniform. The result that there is no contribution to M from parts of C which coincide with radial lines under a variety of conditions can be used to great advantage, as will be demonstrated.

Next, the contribution to M from a part of C which coincides with a segment of a circle centered at the origin is considered, with particular emphasis on circular segments of infinitely large or infinitesimally small radius. It is well-known that the plane elastic stresses due to a concentrated force acting at the origin in an unbounded body or at the apex of a wedge vary with radial distance $r = (x_1^2 + x_2^2)^{1/2}$ as r^{-1} for any fixed value of angle $\theta = \tan^{-1}(x_2/x_1)$. The strain energy density W therefore varies with radial distance as r^{-2}. On the other hand, for an increment $d\theta$ in the angle θ, the term $x_i n_i \, ds$ which multiplies W in the integrand of the M-integral is equal to $r^2 \, d\theta$ on a circular segment of radius r. If a similar argument is applied to the second term in the integrand of M, then it becomes clear that the integrand is independent of r for a concentrated force in an unbounded body or at the apex of a wedge, and the evaluation of the contribution to M from a circular arc in this case amounts to integration of the known angular variation of the elastic fields over some appropriate range of angle θ. It is noteworthy that such a contribution is well-defined even when the arc has an infinitely large or an infinitesimally small radius. Thus, with a view toward applying the M-integral in the analysis of elastic crack problems, if parts of the path C are circular arcs along which the elastic field is indistinguishable from the *known* elastic field of a concentrated force then the contribution to M from these parts of C can be determined without solving the elastic crack boundary value problem. It is known that the dependence of the stresses on radial distance r for a discrete elastic dislocation at the origin is also r^{-1}. Therefore, if parts of C are circular arcs along which the elastic field is indistinguishable from that due to a discrete dislocation, then contributions to M from these parts of C can also be directly determined.

In view of the special properties of the M-integral which were cited above, the procedure for applying the conservation law $M = 0$ to determine elastic crack tip stress intensity factors should be clear. For the specific problems to be considered, closed contours C can be found which are entirely made up of radial lines along which the integrand of (1) is zero, circular arcs along which the elastic fields are essentially those of a concentrated force or discrete dislocation, and vanishingly small paths around the crack tip which yield a contribution proportional to the energy release rate G. Under suitable circumstances, the stress intensity factor corresponding to this value of G may be determined according to (3). The procedure is next indicated for several such problems.

REPRESENTATIVE STRESS INTENSITY FACTOR CALCULATIONS

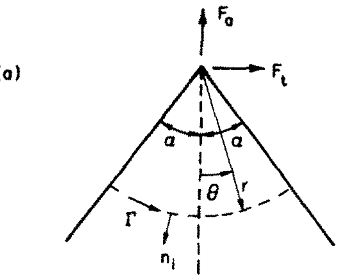
A result which is of use in analyzing many plane problems of the type being considered here is the value of the M-integral for a circular arc in a wedge loaded by a concentrated force at its apex, and this result is derived for the general configuration before specific problems are examined. Consider a wedge of apex angle 2α which is subjected to an axial force F_a and a transverse force F_t at its apex, as shown in Fig. 1(a). As is shown in the book by Timoshenko and Goodier[7], the resulting in-plane stress components referred to polar coordinates (r, θ) are

$$\sigma_{rr} = \frac{2F_a \cos\theta}{r(2\alpha + \sin 2\alpha)} - \frac{2F_t \sin\theta}{r(2\alpha - \sin 2\alpha)} \tag{4}$$

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0. \tag{5}$$

For wedge deformation under plane strain conditions without rigid body rotation, the integrand of M corresponding to the stress state[4, 5] is

$$Wx_i n_i - T_k u_{k,i} x_i = -\frac{1}{2}\frac{(1-\nu^2)}{E} r\sigma_{rr}^2 \tag{6}$$



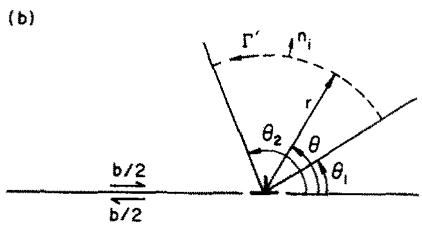


Fig. 1. The corner loaded elastic wedge and a straight edge dislocation are shown in (a) and (b), respectively, and the corresponding values of M along Γ and Γ' are given in eqns (7) and (11).

on the circular arc Γ of radius r in Fig. 1(a). The result of integrating (6) on Γ for $-\alpha \leq \theta \leq \alpha$ is

$$M(\Gamma) = -\frac{1-\nu^2}{E}\left\{\frac{F_a^2}{(2\alpha+\sin 2\alpha)}+\frac{F_t^2}{(2\alpha-\sin 2\alpha)}\right\}. \tag{7}$$

A second general result which is of use in analyzing a number of problems is the value of the M-integral for a circular arc centered at an edge dislocation, as shown in Fig. 1(b). The Burger's displacement of the edge dislocation is b with the sense shown and the circular arc Γ' has radius r and angular range $\theta_1 \leq \theta \leq \theta_2$. The in-plane stress components referred to the polar coordinates (r, θ) are

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{Eb\sin\theta}{4\pi(1-\nu^2)r} \tag{8}$$

$$\sigma_{r\theta} = \frac{Eb\cos\theta}{4\pi(1-\nu^2)r}. \tag{9}$$

The integrand of M corresponding to the stress state (8, 9) and plane strain conditions is

$$Wx_in_i - T_ku_{k,i}x_i = \frac{Eb^2\cos^2\theta}{8\pi^2(1-\nu^2)r} \tag{10}$$

on the circular arc Γ' of radius r, and the result of integrating (10) on Γ' for $\theta_1 \leq \theta \leq \theta_2$ is

$$M(\Gamma') = \frac{Eb^2}{32\pi^2(1-\nu^2)}[2\theta_2 + \sin 2\theta_2 - 2\theta_1 - \sin 2\theta_1]. \tag{11}$$

With the aid of the general results in (7) and (11), a number of stress intensity factor solutions may now be written down without further calculation.

Demonstration of procedure

Four plane elastic crack problems are represented by the sketches in Fig. 2. The crack tip stress intensity factors for these problems are known exactly, of course, and they may all be found in [8] where original source references are given. However, each of the problems

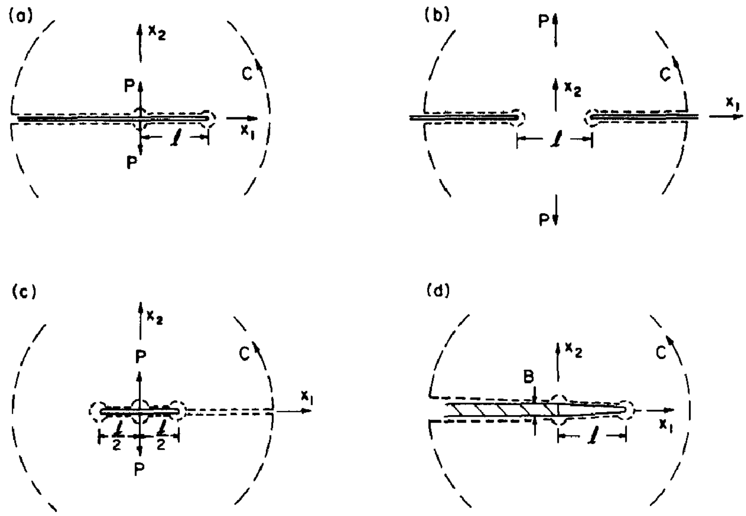


Fig. 2. Some standard elastic crack problems for which the conservation law $M(C)=0$ leads directly to stress intensity factor results.

represented in Fig. 2 is such that the stress intensity factor can be calculated simply and directly by application of the conservation law $M=0$, and it would seem worthwhile to first demonstrate the approach for these familiar problems. In each case, a suitable closed contour C is shown, where it is assumed that the radii of the small and large circular arcs of C are arbitrarily small and large, respectively, as compared to the characteristic length l in each problem.

For the semi-infinite crack with point loads acting on the crack faces in Fig. 2(a), there is no contribution to M from the straight parts of C along the crack faces because these are radial segments on which traction is zero. Furthermore, there is no contribution from the infinitely large circle because the applied loads are self-equilibrating and therefore the stress components are $o(r^{-1})$ as $r\to\infty$ for this configuration. It is reasonable to expect that the stress state arbitrarily close to the load points will be indistinguishable from the stress state in a half-plane (i.e. a wedge with $\alpha=\pi/2$) with a normal boundary load. If the general result (7) is applied for the small arcs around the load points with $F_a=P$, $F_t=0$ and $\alpha=\pi/2$ and if due account is taken of the sense of the normal n_i, then

$$M(C)=2\frac{1-\nu^2}{E}\frac{P^2}{\pi}-lG=0 \tag{12}$$

and, from Irwin's relationship (3), the well-known mode I stress intensity factor result is obtained, that is, $K_I=P\sqrt{(2/\pi l)}$. The equivalent mode II problem with tangential crack face loads may be analyzed in much the same way.

A ligament between two semi-infinite coplanar cracks which carries a total force P is shown in Fig. 2(b). The force P in the ligament arises from the application at remote points of the body of surface tractions or body forces with resultant force P in the direction shown and zero resultant moment. If it is assumed that the stress distribution in the ligament depends on the resultant of the remotely applied tractions or body forces but is independent of the remote traction or body force distribution, then the remote elastic field may be viewed as being that due to a normal concentrated force of magnitude P acting on the edge of an otherwise traction free half-plane. As before, there is no contribution to M from the parts of C along the crack faces, and the remote elastic field is that of a normal point force acting on the edge of a half-plane. If the result (7) is applied for the remote contour with $F_a=P$, $F_t=0$ and $\alpha=\pi/2$, then $M(C)=0$ implies

$$-2\frac{1-\nu^2}{E}\frac{P^2}{\pi}+2\frac{l}{2}G=0 \tag{13}$$

which leads immediately to the known stress intensity factor result $K_I = P\surd(2/\pi l)$. The analysis of the symmetrically point loaded finite crack shown in Fig. 2(c) involves steps virtually identical to those followed in the preceding two examples, and all details are omitted. The stress intensity factor for this case is $K_I = P\surd(2/\pi l)$.

The wedged-open crack model shown in Fig. 2(d) has played a central role in the development of cleavage fracture initiation models. Contributions to the value of M from parts of C along the crack faces vanish because these are radial lines on which the traction is zero. Also, contributions to M from parts of C along the wedged-open section vanish because these are radial lines on which the tangential component of traction is zero and the normal component of displacement is uniform. The internal stresses at the left end of the crack are expected to be square-root singular but, as noted by Eshelby[2], if the origin of coordinates is located at this point, the factor x_i in the integrand of M renders the integrand nonsingular and there is no contribution to M from the small arcs at the origin. For the remote contour, the general result (11) may be applied with $b = B$, $\theta_1 = -3\pi/2$ and $\theta_2 = \pi/2$. Application of the conservation law $M = 0$ for the path C in Fig. 2(d) leads to the relation

$$\frac{EB^2}{8\pi(1-\nu^2)} - lG = 0, \tag{14}$$

and the corresponding stress intensity factor is $K_I = BE/[(1-\nu^2)2\surd(2\pi l)]$.

Further applications of $M = 0$

The four plane elastic crack problems represented in Fig. 3 are of the type which can be analyzed by applying the conservation law $M = 0$. Consideration of the corner loaded edge crack shown in Fig. 3(a) was undertaken in an attempt to explain a particular phenomenon in indentation fracture. It is well-known that if a hard indenter is pressed normally into the surface of a brittle solid then cracks form near the indenter, with different types of fractures occurring for different indenter configurations and material characteristics. Of particular interest here are certain cracks which open along planes which are normal to the solid surface and which contain the indenter axis, that is, the so-called median planes. In a report on a study of indentation fracture of cemented carbides[9], it was noted that some of these cracks experience greater growth during *reduction* of the normal indentation force than during the increase of the indentation force to its maximum value. In Fig. 3(a), the indentation force is represented by the normal force of magnitude $2P$, and the opposed tangential forces of magnitude Q are included

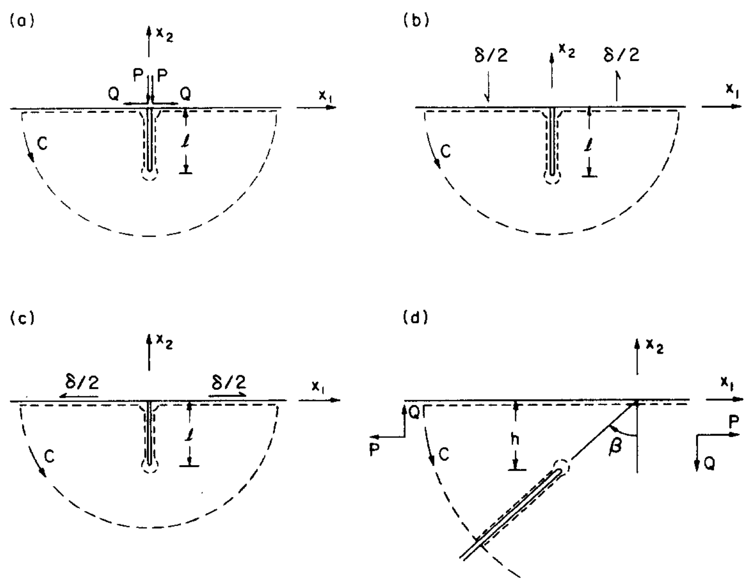


Fig. 3. Additional elastic crack problems which can be analyzed by means of the conservation law $M(C) = 0$.

to represent the tendency for the inelastically deforming material directly beneath the indenter to flow outward.

The stress intensity factor for this problem is determined by direct application of $M = 0$ for the contour C shown in the sketch. To compute the contribution to M from the small arcs around the corner points $x_1 = \pm 0$, $x_2 = 0$, the general result (7) with $\alpha = \pi/4$, $F_a = -(P+Q)/\sqrt{2}$ and $F_t = \mp(P-Q)/\sqrt{2}$ is employed. Likewise, the contribution to M from the large arc centered at the origin is given by (7) with $\alpha = \pi/2$, $F_a = -2P$ and $F_t = 0$. The only other contribution to M comes from the vanishingly small path surrounding the crack tip. If the energy release rate is expressed in terms of the mode I stress intensity factor according to (3) then application of the conservation law yields the result that

$$K_I = \left\{\frac{4\pi}{l(\pi^2-4)}\right\}^{1/2}(Q - 2P/\pi). \tag{15}$$

If (15) is considered in terms of possible load paths in the (P, Q)-plane then some interesting results emerge (see Fig. 4). It is clear that $K_I\sqrt{l} = 0$ along the line $Q = 2P/\pi$ and, in fact, $K_I\sqrt{l}$ is constant along any line with slope $2/\pi$. If the indenter geometry and material characteristics are such that indentation is represented by load trajectory OA, then the load point continuously moves in a direction of *decreasing* $K_I\sqrt{l}$, which suggests that the initiation and growth of an indentation fracture is unlikely. On the other hand, if the geometry and material properties are such that indentation is represented by load trajectory OB, then the load point continuously moves in a direction of *increasing* $K_I\sqrt{l}$. If it is assumed that crack growth occurs at a constant value of K_I, say K_{Ic}, then loading along OB is accompanied by crack growth from zero initial crack length, for example. If the variation of Q during reduction of the indentation force P is such that unloading occurs along BO, then again the load point moves in a direction of decreasing $K_I\sqrt{l}$, which suggests a reduction in K_I below K_{Ic} and no further crack growth. However, if the material properties and geometrical configuration are such that unloading (i.e. reduction in P) occurs along BC then the load point continues to move in a direction of *increasing* $K_I\sqrt{l}$, which suggests that the crack could continue to grow as the indentation force is reduced. While this model is consistent with the observation of indentation crack growth during both increase and decrease of the indentation force under suitable conditions, it is far from being a complete analysis of the phenomenon. Such an analysis would necessarily include a determination of the dependence of Q on the indentation force, the geometry of the indenter, and the elastic–plastic properties of the material.

A second problem which arises in the study of contact fracture is represented in Fig. 3(b). It is assumed that a rigid surface with an irregularity in the form of a simple step of height δ is pressed against the initially plane surface of a solid. The resulting stress concentration at the step is relieved by growth of a crack as shown. If it is assumed that the material is in contact with the stepped indenter all along its edge $x_2 = 0$ and that the indenter exerts no shear traction on the edge $x_2 = 0$ then the dependence of the mode II stress intensity factor K_{II} on step height

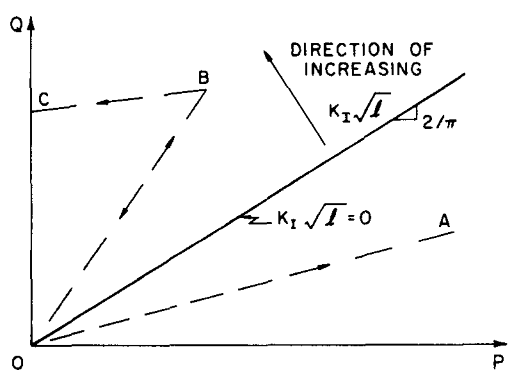


Fig. 4. Possible loading and unloading paths for the corner loaded edge crack in Fig. 3(a). As discussed in the text, crack growth is possible for both increasing and decreasing normal force P, depending on material characteristics.

δ may be determined from $M = 0$ for the path C shown. The contribution to M from the large circular arc is given by (11) with $\theta_1 = -3\pi/2$, $\theta_2 = -\pi/2$ and $b = 2\delta$, and the only other contribution to M arises from the small crack tip contour. In view of (3), the result of applying $M = 0$ to this problem is

$$K_{II} = \frac{E\delta}{2(1-\nu^2)\sqrt{(\pi l)}}. \tag{16}$$

A similar edge-crack problem involving imposed displacements is represented in Fig. 3(c). Such a crack might be viewed as being pryed open in a very stiff loading system or, on the microscale, it is a standard example of a microcrack nucleated by a dislocation pile-up mechanism (e.g. see pp. 29–31 of [10]). If the edge $x_2 = 0$ is free of normal traction and the part of the edge $x_1 > 0$ is displaced uniformly an amount δ with respect to the part $x_1 < 0$, then application of $M = 0$ yields a mode I stress intensity factor identical in form to (16), that is, $K_I = E\delta/[2(1-\nu^2)\sqrt{(\pi l)}]$.

Finally, the plane problem of a semi-infinite crack in a half-plane with traction free surface represented in Fig. 3(d) is considered. The crack is inclined by an angle β to the free surface normal, and the remaining ligament of thickness $h = l \cos\beta$ transmits a tensile force P and a shear force Q. In this context, P and Q are called tensile and shear forces, respectively, because of their orientation with respect to the free edge $x_2 = 0$. In general, this is a mixed mode crack problem and, although the energy released per unit coplanar crack extension G may be calculated by application of the conservation law, the separate stress intensity factors K_I and K_{II} cannot be deduced. However, certain crack growth criteria for biaxial loading conditions rely on a knowledge of the variation of energy release rate with direction of crack growth, so that the calculation of G in terms of β for arbitrary P and Q would seem to be worthwhile. The contributions to M from the large arcs $0 \le \theta < \beta + \pi/2$ and $\beta + \pi/2 < \theta \le \pi$, respectively, are given by (7) with $\alpha = \pm(\beta \pm \pi/2)/2$, $F_a = P\cos\alpha \pm Q\sin\alpha$ and $F_t = \mp P\sin\alpha + Q\cos\alpha$. The dependence of energy release rate G on β follows from application of $M = 0$ and is given by

$$Gh = \frac{1-\nu^2}{2E}\cos\beta\left\{\frac{A+B}{[(\pi/2)-\beta+\cos\beta]} + \frac{A-B}{[(\pi/2)-\beta-\cos\beta]} + \frac{A-B}{[(\pi/2)+\beta+\cos\beta]} + \frac{A+B}{[(\pi/2)+\beta-\cos\beta]}\right\} \tag{17}$$

where

$$A = P^2 + Q^2, \quad B = (P^2 - Q^2)\sin\beta - 2PQ\cos\beta. \tag{18}$$

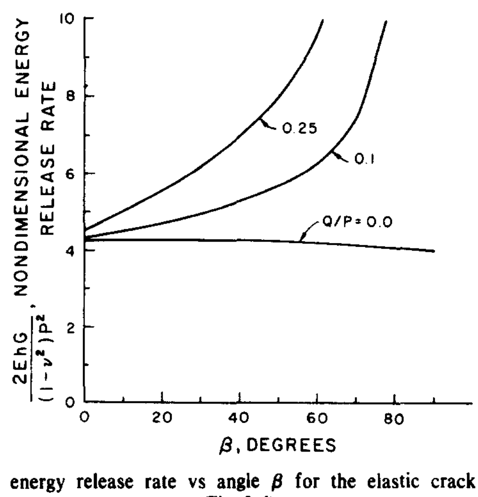


Fig. 5. Nondimensional energy release rate vs angle β for the elastic crack problem represented in Fig. 3(d).

It is easily verified that for the special case $\beta = 0$, $Q = 0$ the result (17) reduces to the stress intensity factor result

$$K_I = P\left\{\frac{4\pi}{(\pi^2 - 4)h}\right\}^{1/2} \tag{19}$$

which was obtained by Stallybrass[11] by direct application of integral transform methods. The equivalent mode II result for $\beta = 0$, $P = 0$ which is reported in [8] may also be extracted from (17). Typical nondimensional graphs of G versus β are shown in Fig. 5. For $Q = 0$, G has a very shallow maximum at $\beta = 0$, while for any $Q > 0$, G becomes infinitely large as $\beta \to \pi/2$. This suggests that if a crack approaches a free boundary in the presence of a shear force then there is a tendency for the direction of crack growth to become parallel to the boundary. Such behavior was reported by Kinra and Kolsky[12] in their experiments on the fracture of brittle plastic beam.

DISCUSSION

There are, of course, many other crack problems which satisfy the main conditions for application ofthe M-integral approach, i.e. a stress singularity of order r^{-1} as $r \to 0$ and/or as $r \to \infty$ and radial boundaries with suitable boundary conditions. Problems were chosen for this discussion simply to illustrate the range of the approach.

It should perhaps be noted that it is not essential that the origin of coordinates be located at a particular singular point of the stress field, as was the case in all of the examples above. If coordinates are chosen such that the singular point of the stress field is at x_i^0, then the analysis proceeds in the same way except that the conservation law $M = 0$ is replaced by $M - x_i^0 J_i = 0$, where J_i is given in (2). This change in form of the conservation law is due only to the coordinate transformation, and the elastic field is still presumed to be non-singular everywhere within the contour C.

Finally, certain inherent limitations of the approach are recalled. First, if the general conditions for application of the method are satisfied but two crack tips are involved, neither of thich is at the origin, then the division of the total energy release rate between the two crack tips cannot be determined. Likewise, under mixed mode I and mode II conditions, the total energy release rate may be determined but the individual stress intensity factors for each mode cannot be established. In the case of problems like that represented in Fig. 3(d) it would be desirable to include rigid rotations at infinity to simulate transfer of bending loads across the unfractured ligament. However, it appears that if such a rotation is included then the moment required to produce this rotation for any given problem must be known. Determination of this moment is equivalent to a complete solution of the problem[13] which defeats the purpose of the present approach.

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